a review of Norm-square localization and the quantization of Hamiltonian loop group spaces. by Loizides, Yiannis; Song, Yanli

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Norm-square localization and the quantization of Hamiltonian loop group spaces. (English)
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J. Funct. Anal. 278, No. 9, Article ID 108445, 45 p. (2020).

Let G be a compact connected Lie group, which is assumed to be simple and simply connected, LG the loop group of G, $\Phi_{\mathcal{M}}:\mathcal{M}\to L_{\mathfrak{g}^*}$ a Hamiltonian LG-space, with level $k>0$ prequantum line bundle L and $T\subset G$ a maximal torus. In ["Spinor modules for Hamiltonian loop group spaces", Preprint, arXiv:1706.07493], the authors, together with *E. Meinrenken*, constructed a finite-dimensional *global transversal* $\mathcal{Y}\subset\mathcal{M}$ as well as a canonical spinor module $S_0\to\mathcal{Y}$. The submanifold $\mathcal{Y}$ is a small *thickening* of the singular subset $\mathcal{X}=\Phi_M^{-1}(t^*)\subset\mathcal{M}$. The submanifold $\mathcal{Y}$ is equipped with a moment map $\phi:\mathcal{Y}\to\mathfrak{t}$, which is proper on the support of the Bott-Thom element.

In [Math. Ann. 374, No. 1–2, 681–722 (2019; Zbl 1416.81068)], the authors investigated a Dirac-type operator $\mathcal{D}$ on $\mathcal{Y}$ acting on sections of $\wedge\mathfrak{n}_-\widehat{\otimes}S$, where $S=S_0\otimes L$ and $\mathfrak{n}_-\subset\mathfrak{g}_{\mathbb{C}}$ denotes the sum of the negative root spaces of G. The operator $\mathcal{D}$ was shown to represent an index pairing between a spin-c Dirac operator for S and the pullback of a Bott-Thom element for $\mathfrak{g}/\mathfrak{t}$, the latter formally playing the role of a Poincaré dual to $\mathcal{X}$ in $\mathcal{Y}$.

This paper, consisting of 6 sections, studies a deformation of the above operator in the vein of [*Y. Tian* and *W. Zhang*, Invent. Math. 132, No. 2, 229–259 (1998; Zbl 0944.53047); *X. Ma* and *W. Zhang*, Acta Math. 212, No. 1, 11–57 (2014; Zbl 1380.53102); *X. Ma* and *W. Zhang*, Prog. Math. 297, 299–315 (2012; Zbl 1270.58013)]. §2 is a brief introduction to elliptic boundary value problems, following mostly [*C. Bär* and *W. Ballmann*, Prog. Math. 319, 43–80 (2016; Zbl 1377.58022); Surv. Differ. Geom. 17, 1–78 (2012; Zbl 1331.58022)]. §3 summarizes some results from [Math. Ann. 374, No. 1–2, 681–722 (2019; Zbl 1416.81068); arXiv:1706.07493]. §4 introduces a deformation $\mathcal{D}_t$ of $\mathcal{D}$ defined by

$$\mathcal{D}_t=\boldsymbol{D}+(1+t)f\theta\widehat{\otimes}1-it\widehat{\otimes}\mathrm{c}(v_{\mathcal{Y}}).$$

Inspired by [*X. Ma* and *W. Zhang*, Acta Math. 212, No. 1, 11–57 (2014; Zbl 1380.53102)], §5 establishes a formula for index$(\mathcal{D})$, which is expressed as a sum of contributions indexed by the components $Z_\beta=\mathcal{Y}^\beta\cap\phi^{-1}(\beta)$ of $Z=\{v_{\mathcal{Y}}=0\}$:

$$\operatorname{index}(\mathcal{D})=\sum_{\beta\in W\cdot\mathcal{B}}\lim_{t\to\infty}\operatorname{index}_{\mathrm{APS}}(\mathcal{D}_t\restriction U_\beta)$$

Each constribution is a limit in $R^{-\infty}(T)$ as $t\to\infty$, of the index of an Atiyah-Patodi-Singer (APS) boundary value problem on a compact neighborhood U_β of $Z_\beta\cap\mathcal{X}$.

Following the lines of [Zbl 1380.53102; Zbl 1270.58013; *M. Braverman*, K-Theory 27, No. 1, 61–101 (2002; Zbl 1020.58020)], §6 establishes a formula for the contributions in the above formula in terms of transversally elliptic operators in the spirit of [*P.-E. Paradan*, J. Funct. Anal. 187, No. 2, 442–509 (2001; Zbl 1001.53062)]:

$$\operatorname{index}(\mathcal{D})=\sum_{\beta\in W\cdot\mathcal{B}}\operatorname{index}(\sigma_{\beta,\theta}\otimes\operatorname{Sym}(\upsilon_\beta))$$

where $\sigma_{\beta,\theta}$ is a transversally elliptic symbol on the fixed-point set $\mathcal{Y}^\beta$, and υ_β is the normal bundle to $\mathcal{Y}^\beta$ in $\mathcal{Y}$. endowed with a β-*polarized* complex structure. The formula is sometimes called a *norm-square localization* formula. §6.4 is concerned with the $[Q,R]=0$ theorem, which follows from the main result of this paper together with a relatively small part of [arXiv:1604.01965], though its complete proof is postponed to a subsequent paper. The most important application of the $[Q,R]=0$ theorem for Hamiltonian loop group spaces is to the Verlinde formula, concerning which the reader is referred to [Zbl 0997.53066] for a symplectic approach and to [*E. Meinrenken*, Contemp. Math. 583, 175–210 (2012; Zbl 1365.53080); *E. Meinrenken*, Int. Math. Res. Not. 2012, No. 20, 4563–4618 (2012; Zbl 1260.53143)] for the relationship with $[Q, R] = 0$ for Hamiltonian loop group spaces and quasi-Hamiltonian G-spaces.

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MSC:

58J20 Index theory and related fixed-point theorems on manifolds
53D50 Geometric quantization
58J32 Boundary value problems on manifolds
53D20 Momentum maps; symplectic reduction

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References:

[1] Alekseev, A.; Malkin, A.; Meinrenken, E., Lie group valued moment maps, J. Differ. Geom., 48, 445-495 (1998) · Zbl 0948.53045

[2] Alekseev, A.; Meinrenken, E.; Woodward, C., The Verlinde formulas as fixed point formulas, J. Symplectic Geom., 1, 1-46 (2001) · Zbl 1033.53074

[3] Atiyah, M. F., K-theory (1967), Benjamin

[4] Atiyah, M. F., Elliptic Operators and Compact Groups, vol. 401 (2006), Springer

[5] Atiyah, M. F.; Patodi, V. K.; Singer, I. M., Spectral asymmetry and Riemannian geometry. I, Math. Proc. Camb. Philos. Soc., 77, 43-69 (1975) · Zbl 0297.58008

[6] Ballmann, W.; Bär, C., Boundary value problems for elliptic differential operators of first order, (Cao, H.-D.; Yau, S.-T., Surv. Diff. Geom., vol. 17 (2012), Int. Press), 1-78 · Zbl 1331.58022

[7] Ballmann, W.; Bär, C., Guide to elliptic boundary value problems for Dirac-type operators, (Arbeitstagung Bonn 2013 (2016), Springer), 43-80 · Zbl 1377.58022

[8] Berline, N.; Getzler, E.; Vergne, M., Heat Kernels and Dirac Operators (1992), Springer-Verlag · Zbl 0744.58001

[9] Berline, N.; Vergne, M., L'indice équivariant des opérateurs transversalement elliptiques, Invent. Math., 124, 1-3, 51-101 (1996) · Zbl 0883.58037

[10] Bismut, J. M.; Labourie, F., Symplectic Geometry and the Verlinde Formulas, Surveys in Differential Geometry: Differential Geometry Inspired by String Theory, 97-311 (1999), Int. Press: Int. Press Boston, MA · Zbl 0997.53066

[11] Bott, R.; Tolman, S.; Weitsman, J., Surjectivity for Hamiltonian loop group spaces, Invent. Math., 155, 225-251 (2004) · Zbl 1067.53067

[12] Braverman, M., Index theorem for equivariant Dirac operators on noncompact manifolds, K-Theory, 27, 1, 61-101 (2002) · Zbl 1020.58020

[13] Cordes, H.; Labrousse, J., The invariance of the index in the metric space of closed operators, J. Math. Mech., 693-719 (1963) · Zbl 0148.12402

[14] Dai, X.; Zhang, W., Splitting of the family index, Commun. Math. Phys., 182, 2, 303-317 (1996) · Zbl 0874.58076

[15] Freed, D. S.; Hopkins, M. J.; Teleman, C., Loop groups and twisted K-theory II, J. Am. Math. Soc., 26, 595-644 (2013) · Zbl 1273.22015

[16] Gilkey, P. B., On the index of geometrical operators for Riemannian manifolds with boundary, Adv. Math., 102, 2, 129-183 (1993) · Zbl 0798.58071

[17] Grisvard, Pierre, Elliptic Problems in Nonsmooth Domains, vol. 69 (2011), SIAM · Zbl 1231.35002

[18] Guillemin, V.; Sternberg, S., Geometric quantization and multiplicities of group representations, Invent. Math., 67, 3, 515-538 (1982) · Zbl 0503.58018

[19] Harada, M.; Karshon, Y., Localization for equivariant cohomology with varying polarization, Commun. Anal. Geom., 20, 869-947 (2012) · Zbl 1277.53085

[20] Higson, N., A primer on KK-theory, (Operator Theory: Operator Algebras and Applications, Part 1 (1990)), 239-283 · Zbl 0718.46052

[21] Hochs, P.; Song, Y., Equivariant indices of Spin-c Dirac operators for proper moment maps, Duke Math. J., 166, 1125-1178 (2017) · Zbl 1370.58010

[22] Kaad, J.; Lesch, M., A local global principle for regular operators in Hilbert \(C^\ast \)-modules, J. Funct. Anal., 262, 10, 4540-4569 (2012) · Zbl 1251.46030

[23] Kac, V., Infinite-Dimensional Lie Algebras (1994), Cam. Univ. Press

[24] Kasparov, G., Equivariant KK-theory and the Novikov conjecture, Invent. Math., 91, 147-201 (1988) · Zbl 0647.46053

[25] Kato, T., Perturbation Theory for Linear Operators (1984), Springer-Verlag

[26] Kirk, P.; Klassen, E., Continuity and analyticity of families of self-adjoint Dirac operators on a manifold with boundary, Ill. J. Math., 42, 1, 123-138 (1998) · Zbl 0962.58010

[27] Kirwan, F., Cohomology of Quotients in Symplectic and Algebraic Geometry (1984), Princeton Univ. Press · Zbl 0553.14020

[28] Kucerovsky, D., A short proof of an index theorem, Proc. Am. Math. Soc., 3729-3736 (2001) · Zbl 0982.58015

[29] Loizides, Y., Geometric K-homology and the Freed-Hopkins-Teleman theorem

[30] Loizides, Y., Norm-square localization for Hamiltonian LG-spaces (2017), Ph.D. Thesis · Zbl 1359.53067

[31] Loizides, Y., Norm-square localization for Hamiltonian LG-spaces, J. Geom. Phys., 114, 420-449 (2017) · Zbl 1359.53067

[32] Y. Loizides, E. Meinrenken, Verlinde sums and \([Q, R] = 0\), in preparation.

[33] Loizides, Y.; Meinrenken, E.; Song, Y., Spinor modules for Hamiltonian loop group spaces, J. Symplectic Geom., 18 (2020)

[34] Loizides, Y.; Song, Y., Quantization of Hamiltonian loop group spaces, Math. Ann., 374, 681-722 (2019) · Zbl 1416.81068

[35] Ma, X.; Zhang, W., Geometric quantization for proper moment maps, C. R. Math., 347, 7-8, 389-394 (2009) · Zbl 1167.53073

[36] Ma, X.; Zhang, W., Transversal index and \(L^2\)-index for manifolds with boundary, (Dai, X.; Rong, X., Metric and Differential Geometry: the Jeff Cheeger Anniversary Volume (2012), Springer), 299-315 · Zbl 1270.58013

[37] Ma, X.; Zhang, W., Geometric quantization for proper moment maps: the Vergne conjecture, Acta Math., 212, 1, 11-57 (2014) · Zbl 1380.53102

[38] Meinrenken, E., Symplectic surgery and the Spin-c Dirac operator, Adv. Math., 134, 240-277 (1998) · Zbl 0929.53045

[39] Meinrenken, E., Lectures on group-valued moment maps and Verlinde formulas, Contemp. Math., 583, 175-210 (2012) · Zbl 1365.53080

[40] Meinrenken, E., Twisted K-homology and group-valued moment maps, Int. Math. Res. Not., 2012, 4563-4618 (2012) · Zbl 1260.53143

[41] Meinrenken, E.; Sjamaar, R., Singular reduction and quantization, Topology, 38, 4, 699-762 (1999) · Zbl 0928.37013

[42] Meinrenken, E.; Woodward, C., Hamiltonian loop group actions and Verlinde factorization, J. Differ. Geom., 50, 417-469 (1998) · Zbl 0949.37031

[43] Nicolaescu, L., On the space of Fredholm selfadjoint operators, An. Ştiinţ. Univ. Iaşi, 53, 209-227 (2007) · Zbl 1212.58015

[44] Paradan, P-E., Localization of the Riemann-Roch character, J. Funct. Anal., 187, 442-509 (2001) · Zbl 1001.53062

[45] Paradan, P.-E., Formal geometric quantization II, Pac. J. Math., 253, 169-212 (2011) · Zbl 1235.53093

[46] Paradan, P.-E.; Vergne, M., Index of transversally elliptic operators, Astérisque, 328, 297-338 (2009) · Zbl 1209.19003

[47] Paradan, P-E.; Vergne, M., Equivariant Dirac operators and differentiable geometric invariant theory, Acta Math., 218, 137-199 (2017) · Zbl 1385.53035

[48] Paradan, P-E.; Vergne, Michele, Witten non abelian localization for equivariant K-theory, and the \([Q, R] = 0\) theorem, Mem. Am. Math. Soc., 261, 1257 (2019) · Zbl 07127346

[49] Pierrot, F., Opérateurs réguliers dans les \(C^\ast \)-modules et structure des \(C^\ast \)-algebres de groupes de Lie semisimples complexes simplement connexes, J. Lie Theory, 16, 4, 651-689 (2006) · Zbl 1152.22006

[50] Pressley, A.; Segal, G. B., Loop Groups (1986), Clarendon Press · Zbl 0618.22011

[51] Reed, M.; Simon, B., Methods of Modern Mathematical Physics, vol. 4: Analysis of Operators (1978), Elsevier

[52] Reed, M.; Simon, B., Methods of Modern Mathematical Physics, vol. 1: Functional Analysis (1980), Academic: Academic New York

[53] Shubin, M. A., Spectral theory of the Schrödinger operators on non-compact manifolds: qualitative results, (Spectral Theory and Geometry. Spectral Theory and Geometry, Edinburgh, 1998 (1999), Cambridge Univ. Press), 226-283 · Zbl 0985.58015

[54] Szenes, A.; Vergne, M., \([Q, R] = 0\) and Kostant partition functions, Enseign. Math., 63, 471-516 (2017) · Zbl 1409.81068

[55] Tian, Y.; Zhang, W., An analytic proof of the geometric quantization conjecture of Guillemin—Sternberg, Invent. Math., 132, 229-259 (1998) · Zbl 0944.53047